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ABSTRACT

This pilot study examined the form or nature of the more highly organized knowledge structures used by successful problem solvers and whether there are experiences that would encourage their development. Previous related research and a theoretical framework are discussed. The small-group problem solving mode was used, with "A Problem with Cards" chosen because it provided contexts and materials through which groups of children working together could construct various representations and use different strategies. One of five homogeneously-grouped grade 6 classes in Princeton Day School, with 15 students, was selected. A profile of each student was developed; two problem solving sessions for which students were divided into groups of 2, 3, and 4 were conducted; and each student was interviewed about the sessions. The protocol results are presented and analyzed in some detail, with a focus on commutativity, inverse, and generalizing. Comments from the interviews are also presented. It was concluded that several problem-solving strategies emerged as important. Group problem solving appeared to be an effective means of helping students develop the ability to analyze and solve problems. An appendix contains the card problem and questions. (MNS)

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STRUCTURE VARIABLES AND MULTIPERSON PROBLEM SOLVING PROTOCOLS

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Introduction

Within mathematics education the importance of gaining greater insights into children's problem solving activities is widely recognized. In a review of the research, Lester (1980) observed that over the decade of the 70's mathematics educators devoted more attention to problem solving than any other single area of inquiry. Further, Lester noted:

> ...there is substantial support for the notion that the ultimate aim of learning mathematics at every level is to be able to solve problems (p.187).

However, problem solving is an extremely complex human activity and as such presents significant difficulties in research design, definition and interpretation. Lester observed a wide range in the quality, focus and methodology in problem solving research, as well as a general lack of agreement as to what problem solving involves--at least in specific terms. Overall, the three major foci of problem solving investigations are the problem, the process and the problem solver. The three elements are, of course, inseparable, and the research described in this paper examined all three elements within a carefully chosen and controlled problem solving context.

With regard to the "problem," the term "task variables" is applied as a descriptor to those studies that seek to determine the relationship between the characteristics of a problem, such as the mathematical content, representation and structure, and the performance of the solver. The importance of task variable research is apparent. Certainly the selection of appropriate problems, those that correspond to the needs of students in terms of content, context and degree of difficulty, is an important aspect of mathematics education.

Problems should also be designed to provide students with opportunities for learning, and perhaps the most significant criticism of task variable research has focused on the problems selected. Lesh and Akerstrom (1981), for example, contend:

> The chief justification for giving instructional attention to word problems is that they presumably help students develop problem-solving abilities that are useful in everyday situations.... (However)...if one looks at everyday situations in which mathematics is used, it becomes obvious that: (1) many of the most important problem types are not at all similar to textbook word problems; and (2) the processes, skills, and understandings that are most important in real situations are seldom stressed by mathematics educators (p.117).

Generally the processes of problem solving are more significant than the product, but investigations into the processes that students use in solving problems are complicated by the fact that the important behaviors are covert. The usual methods of gathering data from which those behaviors may be inferred include retrospection at the conclusion of the problem solving session or having the subject "think aloud" during the session. Both techniques have inherent drawbacks. Nisbett and Wilson (1977) argued that individuals do not have direct access to mental processes, but instead report their judgments regarding their responses to problem solving stimuli. Perhaps, then, subjects are reporting imperfectly what they believe occurred or was occurring. On the other hand, Ericsson and Simon (1980) studied how well individuals reported information they were processing in short term memory and concluded that verbal reports, either during or after an exercise, can be a valuable source of data provided care is taken in interpretation.

Recognizing that having subjects discuss their problem solving activities with the researcher provides but a trace of the solution process, some investigators (Schoenfeld, 1982 and Noddings, Chaffe and Enright, 1981) have experimented with small group protocols. In small groups, with

little or no researcher intervention, as subjects discuss the solution of the problem among themselves, they provide direct insights into the cognitive processes employed.

Many of the studies that have focused on the problem solver have attempted to determine the characteristics that differentiate good problem solvers from the less proficient. In some cases a specific listing of the abilities that describe good problem solvers has been proposed. Krutetskii (1976), for example, found that a good problem solver possessed: (1) the ability to distinguish relevant from irrelevant data, (2) the ability to grasp quickly the mathematical structure of the problem, (3) the ability to generalize, and (4) the ability to remember the problem's structure for a long time. Other studies have concentrated on other aspects of the solver. Witkin, Moore, Goodenough and Cox (1977) found that students whose cognitive structure was field independent, that is students who were able to isolate information from an organized field, were more successful on problems that required pertinent information to be extracted than students with a field dependent cognitive structure.

On the basis of a study that required students to sort mathematical word problems, Silver (1979) generally concurred with Krutetskii's description of the abilities possessed by successful problem solvers. Specifically he

found that individuals who sorted problems in terms of mathematical structure rather than context, question form or pseudostructure were the more competent problem solvers. Silver's work is especially interesting, since it investigated the student's (both the proficient and the not so proficient) perception of mathematical structures within problems.

Chi, Glaser and Rees (1981) investigated the characteristics of expert and novice physics problem solvers. The chief distinguishing factor they found was that the experts had significantly more elaborate knowledge structures. In one phase of the study the experts and the novices were asked to sort problems.

> An alternative but consistent interpretation of the Sorting Studies is that experts and novices organize their knowledge in different ways. Experts possess schemas of principles that may subsume schemas of objects, whereas novices may possess only schemas of objects (p.85).

Thus the experimental evidence cited previously indicates that successful problem solvers possess more highly organized knowledge structures than do less successful solvers. The questions that arise naturally are concerned with the form or nature of those structures and whether there are experiences that would encourage their development.

Theoretical Framework

Beyond the routine use of memorized algorithmic techniques, we hypothesize that success in problem solving requires that students understand the mathematical structures that underlie the problem, and based upon that understanding, employ strategies derived from the appropriate cognitive structure. In this context, a cognitive structure is a scheme for dealing with problems of a given category or class. Selection of the appropriate scheme arises from the solver's recognition, on the basis of experience, that the problem elements and their mode of occurrence are common to a particular category or class of problem. In essence, a scheme is a plan for dealing with a class of problems. After recognition, newly encountered problems are appropriately dealt with through the scheme that encompasses the problem class.

Mathematical structures are analogous to cognitive structures in that they contain a planned response to the recognition that a set of relationships falls into a particular class, and thus a given problem may be dealt with successfully by means of mathematical strategies developed for that problem class. The significant difference between cognitive and mathematical structures is that the former are individually developed, often with teaching assistance, while the latter are adopted from educational

experiences. (If they were not, students would have to reinvent mathematics each time.)

Successful problem solvers are those who recognise that a problem falls into a class for which the underlying mathematical structure has been , or may be, accommodated within an internal cognitive structure. Upon recognition, the successful solver employs strategies based on the mathematical structure of the problem in accordance with plans derived from the cognitive structure.

The perspective here is consistent with that of both Piaget and Bruner. Cognitive structures are developmental in nature, and with development levels of complexity are added as connections are established between and among the structural components. According to Bruner and Kenney (1965),

> We would suggest that learning mathematics may be viewed as a microcosm of intellectual development. It begins with instrumental activity, a kind of definition by doing. Such operations become represented and summarized in the form of particular images. Finally, and with the help of a symbolic notation that remains invariant across transformations in imagery, the learner comes to grasp the formal or abstract properties of the things he is dealing with (p.59).

The development of cognitive structures increases the problem solvers' depth of understanding of the related mathematical structures, and the process of problem solving,

in its broadest sense, consists of seeking connections between new experiences and already mastered experiences through the use of internal cognitive structures. The more developed their cognitive structures, the more likely problem solvers are to accommodate mathematical structures within the appropriate cognitive structures. Or, in other words, an individual's ability to generalize from one situation to another is related to the development and elaboration of internal cognitive structures.

One hypothesis of the research discussed in this report is that problem solving proficiency is directly related to the development of cognitive structures that are related to certain fundamental mathematical structures. Earlier research at Rutgers, (Maher, O'Brien, Heller, Smith, Normandia and DeStefano, 1982), has demonstrated that the group properties collectively constitute one extremely important structure. Piaget (1970) also noted the importance of the group concept within the structure of mathematics.

> Groups are today the foundation of algebra. The range and fruitfulness of the notion are extraordinary. We run into it in practically every area of mathematics and logic... Clearly, then, we should try to understand the reasons for the immense success of the group concept. Since groups may be viewed as a kind of prototype of structures in general, and since they are defined and used in a domain where every assertion is subject to demonstration, we must look to them to ground our hope for the future of structuralism (p.19).

The hypothesis governing the research described in this report is that success in problem solving is related to the level of cognitive understanding of the relationships within the problem. Further, mathematical learning occurs best in an environment that provides students with a rich assortment of experiences or problems that focus on the underlying mathematical structures from a variety of perspectives, both concrete and abstract. Problems that encourage a variety of modes of operation and solution, offer students opportunities to recognize the mathematical structures that determine the relationships among the elements of a problem.

The underlying structure of a mathematical problem is the nature of the relationships among its parts. These relationships may be seen from different, but mutually consistent, perspectives. A particular perspective may suggest a unique approach to problem solution. On this basis, correct solutions to the same problem may be arrived at by different strategies.

For instance, within a problem in which the various parts are given to be in proportion to each other, it is possible to consider additive relationship as well as the multiplicative one which describes the relationship most completely. Thus, there are different models which a person may use to represent the relationships within a given problem and there are certainly alternative means of solution of any given problem.

Objectives

In recent years researchers have devised experiments that involved observing small groups of subjects engaged in problem solving activities. In some instances the primary purpose of the research was to observe the decision-making process (Schoenfeld, 1982); in other studies social interaction was an important factor (Noddings, et al 1981). In designing the experiment described in this paper, the small group problem solving mode was selected for two major purposes: (1) careful monitoring and analysis of the statements and actions of the subjects would provide insights into their problem solving processes; and (2) dynamic interactions within the group would promote learning for individual group members.

With this conceptual framework in mind, problems were selected that provided contexts and materials through which groups of children working together could construct various representations and use different strategies to explore the nature of the questions concerning certain properties of an Abelian mathematical group. Further, the children were asked to examine the same questions in terms of numbers. The term "representation of a problem" refers, in accordance with Kulm (1979) to the "...arrangement of and mathematical relationships among all elements of the problem (p.18)" as it is stated by the child and the "...problem solving

strategies (p.18)" intrinsic to the child's construction of the representation.

The objectives of the small group problem solving experiment described in this paper were:

1. To determine the extent to which subjects assisted one another or the group toward the solution of the problem.
2. To note the degree to which the process was a group rather than an individual effort.
3. To determine if the heuristics used by individuals influenced the strategies employed by other members of the group.
4. To observe the modes, concrete and abstract, used in solving the problem.
5. To observe the degree to which the subjects were able afterward to describe their participation in the problem solving process.

Sample

The pilot study discussed in this report was conducted during April and May of 1982 with one of five homogeneously grouped sections of the sixth grade class at Princeton Day School. Based primarily on prior achievement in mathematics, the 15 student section selected for the study was ranked fourth among the five sections.

Princeton Day School is an academically demanding K through 12 independent school of approximately 1,000 students. The majority of the students are from families that would be described as from the middle to upper middle socioeconomic strata. A university town, Princeton has a diverse population, but a long standing tradition of emphasizing academic achievement. Princeton Day School reflects that emphasis.

Methods and Procedures

There were three distinct parts to the pilot study: (1) a profile of each student including assessments of various cognitive structures was developed, (2) two problem solving sessions for which the students were divided into groups of two, three and four were conducted, and (3) interviews with each student in which he or she was asked questions about the problem solving session were held.

Part 1

In addition to the considerable background data available on each student (as collected by the school), clinical tasks designed to assess cognitive structures were administered to each student. All clinical tasks were administered individually to the subjects by clinicians who had received

extensive prior training. The training included the observation of a potential clinician interviewing a test subject by at least two qualified clinicians. Each observer independently recorded the test subject's performance on the task, and at the conclusion of the task the responses and final ratings of the observers were compared and discussed. A trainee was qualified as a clinician only when his or her ratings and responses agreed with those of the experienced clinicians. The subsequent clinical interviews of the subjects who participated in the pilot study were conducted in accordance with carefully developed scripts. In addition to the observational worksheets recorded by the clinicians, audio tapes of the interviews were also made.

In accordance with the theoretical framework of the research, the most important structural variable assessed prior to the problem solving sessions was the subject's understanding of the properties of a group. The task used to assess this structural variable was developed at Rutgers University and discussed in an earlier report (Maher et al, 1982). Briefly, however, the task involved the manipulation of two dolls in accordance with a set of four possible directions for movement. One movement followed by another constituted a completed operation, and the subjects were required to complete a table which exhausted all possibilities of a first move followed by a second move. The

subjects were then guided through a series of questions that explored on the basis of the doll's moves the structure of the properties of an Abelian group; i.e. closure, commutativity, the identity element, the inverse element, and associativity.

Levels of understanding of each of these properties were assessed by a hierarchic series of questions beginning with the most general. Students unable to respond to the general question were provided with "lower level" questions designed to direct attention to specific examples.

Other clinical tasks administered to the subjects included the balance beam task, which according to Piaget and Inhelder (1958) requires formal thought for completion, and an electrical switching task, equivalent to Piaget's chemical task (DeLuca, 1977) that determined each students' development of combinatoric schemes. In addition to the information obtained from the clinical tasks, background data collected by the school were also available.

The clinical interviews were conducted over several weeks during school hours at Princeton Day School using private conference rooms. Additionally, one extended afternoon session was conducted at a nearby public high school when a sufficient number of qualified clinicians were available to interview all of the children individually. The completed protocols for each task were

scored in accordance with the criteria developed for assessment. Finally a profile was developed for each subject with particular emphasis on the subject's performance on the group properties task.

Part 2

Two problems were developed for administration in two 45-minute class sessions one week apart. The problems were adapted from activities described in the School Mathematics Project, Level D (1980).

The 15 subjects were divided into five groups: one of two boys; one of three boys; one of two boys and a girl; one of two girls and a boy; and, one of two boys and two girls. In determining group assignment, consideration was given to the performance of each subject on the group properties task; subjects who had responded at similar levels of understanding were grouped together.

Orientation to Working in Groups

On the day preceding the assignment of the first problem, one of the principal researchers discussed the "rules" governing the problem solving sessions with the entire group. Each problem solving group was to designate one member as the official recorder, but all members of the group were to agree on each conclusion before an answer to a question was to be recorded.

During the actual problem solving sessions, each group was assigned its own space, either a classroom or a conference room. The sessions were audiotaped and one was also videotaped. One or two clinicians attended each session, but the clinicians' role was carefully defined as an observer who was instructed to intercede only if clarification of the instructions became necessary.

At the beginning of each session, the observers gave each subject a copy of the problem. Each group also had two sets of the cards that contained representations of the elements of the set under consideration with which they were to test out solutions. Each student was provided paper and pencil for notes and the observer answered any procedural questions, assured the group that this was not a "graded" event but rather a series of questions to be explored together and reminded them of the need for consensus and a careful recording of each result by the chosen recorder.

For this report, the transcripts of the first of these sessions are discussed. In order to place the follower's discussion in proper context a description of the problem is in order.

The Problem

The problem, entitled "A Problem With Cards," was selected because it provides a concrete model for exploring

the properties of a set of nonnumerical elements with a binary operation that exhibits some but not all of the properties of an Abelian group. The elements of the set consist of five polygonal shapes cut from index cards labelled A, B, C, D and E. For discussion purposes we shall refer to the elements of the set as {A, B, C, D and E}, but in reality the elements are the geometric shapes. The problem script and an illustration of the five index cards is included as an Appendix to this report.

{A, B, C, and D} form a subset of the set of cards which was to be used for the first part of the problem. The shape cut from each of these cards is a rectangle, each different from the other three. The operation consists of placing
one card on top of a second and the result of the operation on any two cards is the resulting shape of the hole formed by the two cards held together.

The first section of the problem explores the property of closure by combining each possible pair of elements under the operation, noting each resulting shape and determining whether it is the same size and shape as the "hole" in one of the original four cards. If so, the name of that resulting card is entered in an appropriate table location. If not, the word "none" is to be written into that cell of the table. The set {A, B, C, D} is closed under the operation.

After closure, two series of questions are posed. The first is to ascertain the existence, if any, of an identity element. If the students decide correctly that the Card D is that element, then the second series is designed to elicit whether there are any pairs of cards which under the operation produce this identity card shape as a result.

The set {A, B, C, D} with the operation defined above is an operational system but, because there are not inverses for each element, it is not a group. The final question in this part of the problem concerns the order of elements: whether the result differs if the second card becomes the first, or top card, and the first becomes the second.

The fifth element of the set, Card E, is an eight-sided polygon which, with other members of the set under the operation, sometimes produces a resulting shape that is the same as one of the original members and sometimes does not. For this reason, the set {A, B, C, D, E} is not closed and therefore is not an operational system. A similar series of questions concerning properties of the set are posed for the full set of five elements as for the first four.

In a final section of the problem, the students are asked questions about identity and inverse elements for the set of "numbers" under the operations addition and multiplication.

Part 3

During the several days following the second problem solving session each student was interviewed by a clinician and asked a series of questions designed to elicit information concerning the following: (1) how much they remembered about both sessions; (2) whether the problems reminded them of other problems, or anything else; and, (3) how they felt about solving such problems with a group.

Analysis of Protocol Results

In light of the objectives of the study, a careful analysis of the transcripts of the audio and video tapes of the groups of children solving the problem was undertaken. Each protocol was examined from three perspectives: (1) instances were noted in which a particular logical or mathematical structure was apparent, either as the focus of a question or as a heuristic used in solving part of the problem; (2) examples of individual's problem solving strategies within the group's consideration of each part of the problem were identified; and (3) occurrences of individual learning resulting from the interaction of the group solving the problem together were noted. To describe and illustrate all three perspectives, excerpts from the protocols will be presented in terms of the particular properties.

Commutativity

Consideration of the property of commutativity within the problem provides an opportunity to illustrate several important issues.

In the four person group, Nils recognizes the symmetry of the table: "Oh, you just have to figure out half the table," and he argues his case even as the others insist on checking each instance.

Nils again: "But they're easy. D and A--A and D. See, you've done it on that side already."

Vince concurs: "They're the same, Uh-huh--," but continues to check his results with the cards.

Nils again insists: "D and B, you see. You can reverse it and do B and D."

Courtney affirms Vince's need to be sure: "If he wants to do it, let him do it."

In response, Nils compares this operation to multiplication: "It's like times. I just realized that when I was half through."

The four children are all convinced, as they work together, of the correct result for each pair of cards taken together.

The next time that the commutative property is considered by this group is when the question of the identity element is posed. Card D has been correctly

chosen as the special card. Nils reads the question: "If this special card is the second card, is the hole that results from placing any first card on top of this special card the same as the first card alone?"

Christine expresses confusion and the two reread the question. Then Nils explains to the others with specific cards: "Look, if you had D and you put that one on it, it has to be A again." Christine then agrees with the conclusion as do the others.

The children have had opportunities to consider examples concretely and to make some generalizations, talking together about the implications of the way the operation transforms elements of the set, before the general question of order is posed.

At that time, Courtney reads the question: "For any two cards, if the order of the cards is reversed and the second card becomes the first card and is placed on top of the other card, what will happen to the resulting hole?"

She immediately responds that: "It will be the same as before."

Nils, however, is unsure and needs to use the cards to consider: "I'm not sure. Look, here's A and D. If you switch--No, it's the same."

Christine initially disagrees with his conclusion but is convinced when he illustrates and says: "Yes it is. Look, if you put it this way, it's still the same hole."

Christine concurs and Courtney makes the generalization: "Oh, so it will always be--"

"The same," Nils finishes.

The members of the group have interpreted the implications of the property to each other and come to agreement. Since each student has participated actively with the cards and in dialogue with the others, it appears that the solution agreed upon has meaning for each. Nils, who frequently jumps to conclusions, has had an opportunity to articulate and test his answer in explanation to the others. Later in the protocol, Christine, who expressed uncertainty several times earlier, is the one who verbalizes the solution.

The question is restated in Part 2 with Card E and the necessity of considering results that are not elements of the original set. Nils is unsure of the commutativity of this set.

Courtney says: "E and A equals <u>that</u> and it's no shape,"

Nils responds with: "Put it backwards."

Christine asserts: "It's the same."

A second group of students, Tucker, Taylor and Laura, and their consideration of the commutative property provides an opportunity for comparison of problem solving

strategies. This group of three was particularly conscientious that the problem solutions be arrived at together. For each part of the problem, they took turns at reading, filling out the table and checking each pair of cards.

Tucker suggested the strategy of filling out the table by rows and the group proceeded until the last row when, at his turn, Tucker states: "O.K. D--Card A on D."

Laura immediately corrects: "No, the other way around." No other member of the group recognizes or notes that the order makes no difference. The issue is not addressed until the question of closure when the three are concerned with the resulting position of several pairs of cards.

When Tucker and Taylor try B on A, Laura again interjects her concern for specific order: "A on top, B on bottom."

They ignore this particular remark and Tucker says: "A on B is C."

Laura emphatically states: "It's B on A."

Tucker repeats: "It's C, we already have that."

Laura responds: "No, we have B on A."

Then Taylor articulates the property: "A on B <u>is</u> B on A--so just draw C--. See, look at Card B and Card A--just switch them over--."

These instances when the results are questioned and discussed perhaps influence Laura's response the next time the question of order is addressed. After all have agreed on D as the identity element, Laura reads the question to the group concerning the special card's being second and immediately responds affirmatively. Tucker rereads the question and Taylor is puzzled by it. Laura never wavers in her certainty.

"It means if you put anything on top of D it will result as the card you're putting on top," explains Tucker.

This time Laura and Taylor chorus: "Yes!"

As with the group of four students, when the general question of order is posed, even though the property has been used in solutions and discussed among the three, the conclusion is not obvious.

Tucker suggests a strategy. "Let's try it. You take two cards--it will stay the same in this one. Let's try some others."

Each member of the group tests out the question concretely, then Laura rereads the question and answers correctly that nothing will change.

For this group, only after all of Part One is completed and they are looking over the table with the E card included does anyone refer to the table's symmetry.

Tucker, after the table is complete, tries to figure out some overall form that might explain logically those results that are not in the original set. His strategy is to look for patterns in the rows.

"Everyone that doesn't have a pattern has a NONE. This one doesn't have a pattern--CCCC NONE. CBCB--This one--ACCA--?" Tucker notes.

Taylor is drawn into this analysis, and comments: "It's not a pattern though--."

Tucker agrees, saying: "Well, ACCA is not really a pattern."

It is then that the two discover the symmetry of the table.

Taylor: "Oh, look!"

Tucker: "You've got the same results down on the different ones as you do across! This is the same as that -."

Taylor responds: "That is the same as that!"

Taking time and interest in this examination of the structure of the problem perhaps influenced their response to the question concerning closure with Card E. They agree that the set is not closed and that E makes the difference.

Taylor now assumes commutativity: "I don't think we have to put both E on A and A on E." Tucker, however, determined to arrive at a thorough conclusion, says:

"Well, I think we should, just to be sure." When Laura gives the pair E on C, Tucker immediately adds C on E.

In the question concerning the existence of an identity element for addition of numbers, it is Laura who now is able to use the property of commutativity in discussion with the others. Tucker reads the question and seems confused about its meaning. He seems to interpret it to be asking for a number in addition that has the property of zero in multiplication.

Laura and Taylor, understanding the question, chorus: "Zero," and Laura adds: "0 plus 4 is 4."

Tucker is still uncertain, thinking aloud: "But it says that is the other number so that would--."

Taylor reads the question aloud again and Laura says: "How about 4 plus 0?"

Taylor agrees and tries to explain to Tucker: "They're talking about either number, Tucker. She is right."

Still Tucker disagrees: "They're talking about is it the same on the second number though--."

Taylor responds to Tucker's meaning of the word second with: "The OTHER number--."

Laura responds with: "0 plus 4 IS 4 plus 0," and when Tucker questions still another time, Taylor combines his argument with Laura's.

"The other number is 4!"

Tucker agrees: "Oh, I was thinking of zero--I was thinking of something else."

It would seem for these children that understanding of the mathematical structure in question is strengthened in several ways. Each has had an opportunity to examine the property within the context of answering specific questions both concretely and in more general forms. The group interaction forced individuals to discuss any lack of understanding and to explain to each other so as to agree on a single solution for each part of the problem. The number of questions and their interrelatedness provided a variety of perspectives from which to explore the meaning of commutativity.

In a third group, the importance of a variety of modes of exploration seems apparent. Melissa, Felicity and Chris, having correctly chosen Card D as the identity card and after completing the table with no reference to symmetry, respond incorrectly to the question of commutativity of the special card.

Felicity reads the general question: "For any two cards, if the order of the cards is reversed and the second card becomes the first card and is placed on top of the other card, what will happen to the resulting hole?"

Melissa immediately responds: "I think it matters."

Felicity insists on trying with the cards. After exploration, first Melissa and then Chris assert: "It doesn't change. It's the same."

When the question is raised for the set of five elements, after they have correctly solved the problems of closure, all three respond emphatically that the order of the two cards has no effect on the result.

The commutative property is used as a strategy for the group of two people. Ashley and Jeremy, both recognize the structure in filling in the table. Ashley says, "A on C. Let's try it."

Jeremy responds: "C on A, you mean."

Ashley rejoins: "C on A. It doesn't matter. It's all the same."

They agree quickly to the general question when it appears.

A final use of the property as a strategy occurs as the two are trying to determine which pairs of cards produce results outside the set of elements. Ashley refers to the property: "There's also NONE for this one."

When Jeremy asks how he knew this to be true, Ashley responds: "Because A on E and E on A are the same thing."

Inverse

The part of the problem that was most difficult for each of the groups was the question posed about inverse elements: "For each card, look for a partner, so that when the partner card is placed on top of the original card, the result is the special card."

Particular perception on the part of the children appears to be needed to agree about the absence of a particular property. The students seemed to expect, if a question were posed, that the element in question would exist. Perhaps fuller understanding concerning a particular concept is reached when students analyze situations in which the structure in question is absent as well as those in which it can be identified.

When Laura first reads this question aloud to her group of three, the consensus response is: "What?"

Tucker responds with the suggestion: "Read it to yourself."

Taylor does so and then, thinking aloud: "It means--Look--Partner cards-- No, it doesn't--"

The three share verbally their uncertainty and move toward a solution.

Tucker posed: "One card is place on one card. The result is D."

The others are not in agreement.

He continues: "There should be a partner for every card."

Taylor tries his hypothesis on the others: "What about this? When you place A on A, it's the result of the special card."

Tucker responds: "No. The special card is D."

Taylor tries once more: "It is the same though."

Tucker rejoins: "It's the result of what you're putting on, but THE result is D."

Taylor then reads the question aloud once more and Tucker joins in the finish with: "The result is the special card--which is D."

Tucker continues: "And there are no two cards when put together will reach D."

Laura, who has listened quietly to the boys asserts: "Except for D."

Each of these children has contributed toward a correct solution.

Several issues seem worthy of consideration. The strategy of reading and rereading in a context of questioning the meaning of the problem appears to have lead this group to a deeper understanding. Taylor's hypothesis that each element was its own inverse was held up to scrutiny by

both boys because of Tucker's care in interpreting the question. Laura, if approaching the problem alone, would perhaps have faltered with her first overwhelmed: "What!" In this context, however, she was included all along. Tucker, who may have reached the same conclusion on his own, seemed to strengthen his understanding and certainty of this concept because he had to convince the others.

Another instance when interaction between children appears to result in deepened understanding occurs in the group of two boys. Jeremy reads the inverse question and then talks it through to himself using specific examples: "So you put B on B--you're gonna get D-- Or B on A--you're gonna get D."

Ashley, after reading the question again to himself, says that he doesn't understand it. Jeremy restates his understanding of the question, again using specific examples but this time trying to explain in such a way that Ashley will understand also. "O.K. What they mean is if you have a B and an A together--you want to look for a partner for B so that it will turn out to be a special card, which is D. So find a card that will make it D--."

Jeremy seems to have understood the meaning of the question and figured out that each pair of cards together should result in D, the identity element. It is in this

second time through, however, in the context of trying to make the idea clear to Ashley that the implications of the statement for the particular set become obvious to him.

He interrupts his explanation to exclaim: "But--it's impossible."

Ashley agrees with him and offers the conclusion: "So the only one that has a partner is D."

When the group of four considers the question of inverse elements some of the same issues arise.

After the first reading Courtney answers that each card is its own inverse.

Christine rejoins: "No," after which each girl rereads the question aloud. Christine then articulates; "So we have to find something that turns out to be D."

Nils, who appears to have been following her reasoning, continues: "You mean two that turn out to be D. D and D."

After this successful choice, Nils begins to try other pairs of cards concretely. He questions: "C onto what turns to D? Let me see a C."

Vince enters the activity to help Nils try out specific pairs of cards. They focus on C and D, perhaps confusing the property desired with that of the identity element.

Vince confesses confusion with: "I don't get this."

Nils continues his trials stating: "That works, too-B and D."

To this comment Christine responds sharply: "Then why isn't it written on the table like that? Look, there's only one D on the whole chart and it's D and D."

Courtney agrees and states the solution somewhat differently: "So that--if they're the only two things that equal D, they're the only two things that can equal D back here."

Nils comes around but on his own terms: "Yeah--so? Hey, look at the chart and see if there's anything equal to D-- All right, only D and D."

The girls tease him a bit and it is interesting to note that, in making sure that Vince agrees with their solution, it is Nils who explains once more: "Look on the chart. See, only D and D make D. None of the others do."

Vince responds: "Right," and records the solution.

Once again a group solution has been reached that has offered each person opportunities to use and articulate different strategies for understanding the syntactic content of the problem as well as its mathematical meaning.

The three who have defended various hypotheses and struggled together for the solution all seem to have gained understanding of the relational structure involved in inverse elements within a set. It would be presumptuous to make such a claim for Vince, but given his participation in

searching for the combinations, Nils' final explanation and the fact that it was he who did record the correct solution, the claim could be made that there was some learning for him as well.

A similar situation occurred in the group of two girls and one boy.

Felicity and Melissa again use the strategy of reading and rereading the question together. Melissa, as before, chooses a concrete example, A, and looks to find a partner card to result in D.

They immediately state the solution. Melissa says: It doesn't work."

Felicity continues: "Write 'none' besides every one but D."

Chris, who is the recorder, then says: "I don't get what they mean by partner card."

There follows a teaching conversation with the two girls explaining and Chris inserting "Yeah" and "Uh-huh" periodically.

This motivation to verbalize carefully and persuasively the solution chosen must certainly reenforce the understanding of the girls, perhaps in a different way than a recitation in response to a teacher's request.

With Chris, as with Vince before, one can only hypothesize that what is gained from participating with the others, requesting an explanation and agreeing to the solution, might lead to some level of understanding of the concept.

Generalizing

In each case the groups of students recognized that the structure of the set of four elements was embedded in that of the set which included Card E.

The first direction in Part 2 of the problem was for a new table to be constructed of the combinations of all five cards. The ways that each group approached the new problem are consistent with the strategies used earlier.

In particular, as Felicity reads the question Melissa talks through the directions with the first pair of cards concretely at hand: "A on A is A."

It is Felicity who then responds: "Wait. It's in the table."

Melissa concurs: "Just copy everything down except for E."

Similarly in the second group of three people, when Tucker reads the question and begins to test out the pairs, Taylor announces: "This is dumb."

Tucker responds: "Let's copy the table until we get to E."

Nils, in the group of four, also sets about retesting all of the cards until the girls refer him to the former table.

Courtney states: "All you have to do is the same--and just add E."

Jeremy and Ashley reach the same conclusion without testing any pairs of cards concretely.

Generalization from understanding of the structure of the first part of the problem occurs frequently as questions are answered about properties in the set of five cards. For example, when the question of the special or identity card is posed, Taylor immediately asserts that it is still D. Tucker agrees and states: "Cause all you have to do is check if it works on E too and it does."

It is interesting to note that each group responds correctly after discussion to the questions about special or identity elements in the set of numbers under multiplication and addition. The conclusions here, however, are consistently made without direct reference to any of the earlier solutions which related to the cards, even though the syntax of the questions are the same.

Strategy for solving this part of the problem did include consideration of specific pairs of numbers. Laura,

in convincing Tucker that zero is the identity for addition, considered 4 and 0.

Melissa and Felicity use 3 and 3000 as examples in testing and explaining to Chris that 1 is the special number for multiplication.

For these children, understanding of the structure of the problem seems to be articulated within each representation. Not yet apparent is an ability to generalize from one representation to another. The single exception was noted earlier when Nils compared the table of cards to multiplication.

These children were also unable to generalize beyond the set of whole numbers in consideration of inverse elements.

Tucker verbalizes understanding that there should logically be partners for numbers other than zero under addition.

Laura, in answer to the question of partner numbers has responded: "Zero plus zero."

Tucker says: "Yes, but they are talking about any number-- That's like saying for 4 is there another number that you add it up gives--"

Taylor and Laura complete the sentence: "Zero."

Taylor says: "I don't get it."

The group goes no further toward a generalization of inverse elements.

Ashley and Jeremy are probed by the clinician to consider the possibility of another pair beyond zero and zero.

Ashley responds: "There isn't--not added--well, I guess there's negative."

Jeremy continues: "But that's so tricky--negative 1, negative 2. But I don't know if you have to have a negative to go backwards."

Felicity, in the interview after the problem was completed, responds that the task did remind her of reciprocals. Her understanding was articulated as "because they turn around." She was unable to give an example of inverses.

Retrospective Comments on the Session

The final phase of the pilot study involved individual interviews with the students in order to sample their reactions to the group problem solving process. It was felt that the student's reactions would provide valuable insights into the pilot study itself as well as suggest improvements to be incorporated into the continuing research. Given the nature of the school (small classes

with class participation highly encouraged), Princeton Day School students are perhaps more verbal than students at other schools, and they have been admitted by means of a selective process. For these reasons no generalizations are possible from the information elicited from the interviews. The children's responses did, however, raise a number of interesting points.

In response to the question as to whether the problem had reminded them of other ideas or activities, several students mentioned the doll rotation task described earlier. Chris also compared the problem to "clock arithmetic" referring to finite sets of numbers under the operations of addition and multiplication which the class had studied earlier in the year.

The children's comments also tended to support the hypothesis that problem solving done by groups working together without teacher intervention provides a rich environment for learning mathematics.

Tucker, for example, stated: "You can find out other people's ideas and see how they match yours."

From a slightly different perspective, Laura noted: "When you have to agree on something, you see the other person's point of view."

This openess to a variety of approaches in the solution

of a problem implies a broader field for understanding the relationships under examination. Interaction within the group involves a critical analysis of one's proposed representation and those of others so that the most appropriate choice can be made from among them.

Thus, according to Felicity: "It's more comfortable. You can talk about it and it helps to figure it out."

Concerning the value of peers as teachers, Chris offers the opinion: "Like when you're not quite sure about it--about the question--they can sort of help you."

Finally, lest we take ourselves too seriously as educators, Tucker asserts: "When it starts to get boring, you have someone to get bored with too."

Conclusions and Implications

On the basis of individual children's responses to parts of the problem, the discussions among children concerning the questions that emerged along the way, and the conclusions that were reached by the various groups, certain implications were suggested as to the way children solve mathematical problems.

As a consequence of the particular problem selected, any specific representational model generated a series of questions about the nature of the property under discussion.

As students went about solving the problem, quite often there was an immediate and spontaneous response from one or more members of the group, and the resulting discussion itself became an important learning mode. Occasionally, there was general concurrence, but more frequently the response had to be explained or defended to the group. In this case, a general understanding of the concept was often expressed as one or more members of the group sought to convince the others. On the other hand, the defense of a proposed response invariably faltered if the sufficiency of the alternative was demonstrated.

Several problem solving strategies emerged as important, both for individuals and groups seeking solutions. The term "strategy" in this context is intended to characterise children's problem solving activities in accordance with Bruner's (1973) interpretation of the term to mean "...a pattern of decisions in the acquisition, retention and utilization of information that seems to meet certain..... objectives (p.135)."

Correct responses to the questions were obtained from an analysis of the table entries that displayed symbolically all possible combinations of the elements in the set. These conclusions emerged without manipulation of the cards; instead they were based upon deductions inferred from the symmetry of the table as well as other patterns noted.

This suggests the achievement of a level of understanding independent of the manipulation of the concrete elements.

Similarly, and often with reference to the table, conclusions to specific questions were arrived at by referring to questions that had been answered in previous sections of the problem. This strategy was particularly evident as the students attempted to answer the questions of Part Two for five elements of the set compared to the four element set of Part One.

Students also employed a strategy of trial and error, sometimes according to a pattern and sometimes randomly, in which specific pairs of cards were used either to prove or disprove guesses or to determine a preliminary response to the question under consideration. This strategy, which probably indicates a lower level of understanding of the logical and mathematical relationships, most often occurred when there was confusion regarding the syntax of a problem, or when one or more of the children expressed a lack of understanding of a concept. The reassurances gained from trying out possible solutions with the cards often led to solutions couched in more general terms.

Based on the examples cited previously, it seems that there is a strong motivation within the group to try to determine the syntactical meaning of each part of the

problem. Whenever one of the students expressed confusion, the question was reread. Sometimes there were three or four readings of a single question accompanied with trials using specific pairs of cards and discussion of the properties being described in order to decipher meanings.

In addition to providing a context in which problem solving strategies were identified, the results obtained by having the children work in small groups appear to have other interesting implications. It is our observation, based on the activities recorded in the protocol transcriptions and supported by the comments of the children in the concluding interviews, that both understanding and confusion about concepts are expressed openly with peers. This tends to support the hypothesis that solving problems in groups is an effective means of helping children develop the ability to analyze problems, develop methods for representing the relationships involved and decide upon means for arriving at solutions. In summary, we believe that allowing students an opportunity to work in small groups on meaningful problems that afford opportunities for uncovering mathematical structures from a variety of perspectives, abstract and concrete, represents an important mode of instruction.

In order to study further the type of problem solving task described in this paper as an effective instrument for

both teaching and research, additional problems need to be developed. The new problems should be part of a carefully constructed experimental framework that includes a profile of the children's knowledge of the concepts under consideration, as well as instruction that is intended to develop an understanding of the concepts in an environment that includes small group problem solving sessions. The experiment should conclude with an assessment of the understanding gained.

The pilot study described in this paper, as well as earlier research conducted at Rutgers University have focused on the group properties as a fundamentally important mathematical structure. Tasks need to be devised that will test the importance of other mathematical structures in terms of students' problem solving abilities.

Appendix

A PROBLEM WITH CARDS

PART ONE - Four Cards

To answer the questions in Part One, you will need the cards marked A, B, C and D

Place Card A on top of Card B. (Make sure that you have not turned either of the cards backwards or upside down). You now have a hole with the same size and shape as the hole in Card C.

Notice that "C" is written in the second box of the first row in the table below.

Fill in the rest of the table with the results of a first card placed on top of a second card. If the result of any first card placed on top of a second card does not match the size or shape of one of the cards, write "NONE" in the appropriate box.

Notice that Card A can be placed on top of another Card A, Card B on top of another Card B and so forth.

FIRST CARD: PLACED ON TOP OF	SECOND CARD: CARD A	CARD B	CARD C	CARD D
CARD A		C		
CARD B				
CARD C				
CARD D			47	

Answer each of these questions using your cards and your table.

1. For every first card placed on top of every second card, is the resulting hole the same size and shape as one of the four original cards?

   Circle: Yes or No

If your answer to question 1 is "No,"

   a. Tell which cards give a different results.

   b. Draw the resulting shapes.

2. Is there a special card which, when placed on top of each one of the cards, does <u>not</u> change the size or shape of the hole in the second card?

   Circle: Yes or No

If your answer to question 2 is "Yes,"

   a. Which one of the cards is the special card?

b. If this special card is the second card, is the hole that results from placing any first card on top of this special card the same as the first card alone?

c. For each card, look for a partner, so that when the partner card is placed on top of the original card the result is the special card.

For each card, if there is a partner card, name it. If there is no partner card, write "NONE" beside that card.

CARD	PARTNER CARD
Card A	
Card B	
Card C	
Card D	

3. For any two cards, if the order of the cards is reversed and the second card becomes the first card and is placed on top of the other card, what will happen to the resulting hole?

For any two cards, does reversing the order change the resulting hole?

Circle: Yes or No

If your answer to question 3 is "Yes,"

a. For which cards is the resulting hole different when the order is reversed?

b. Draw the resulting shapes.

PART TWO - Five Cards

Now add Card E to your other four cards and fill out the table below for all five cards.

FIRST CARD: PLACED ON TOP OF	SECOND CARD: CARD A	CARD B	CARD C	CARD D	CARD E
CARD A					
CARD B					
CARD C					
CARD D					
CARD E					

Answer these questions.

1. For every first card placed on top of every second card, is the resulting hole the same size and shape as one of the original five cards?

If your answer to question 1 is "No,"

a. Tell which cards give a different result.

b. Draw the way the resulting two cards would look (or describe in words).

2. For any two of these five cards, does reversing the order change the resulting hole?

If your answer to question 2 is "Yes,"

a. For which cards is the resulting hole different when the order is reversed?

b. Draw the resulting shapes.

3. For these five cards, is there a special card which, when placed on top of each one of the cards, does <u>not</u> change the size or shape of the hole in the second card?

If your answer to question 3 is "Yes,"

a. Which one of the cards is the special card?

PART THREE - Some Questions About Numbers

1. For numbers, is there a special number which, when you add it to any other number, the answer is that other number?

If your answer to question 1 is "Yes,"

a. What is that number?

b. Is there more than one number that is special in that way?

c. For any number, is there a partner number that adds with the number to give your special number?

d. If you said, "Yes" to c., give an example.

2. For numbers again, is there a special number which, when you multiply it with any other number, the answer is that other number?

If your answer to question 1 is "Yes,"

a. What is that number?

b. Is there more than one number that is special in that way?

c. For any number, is there a partner number that multiplies with the number to give the special number?

d. If you said, "Yes" to c., give an example.

PART FOUR

Try to think of a new set of five cards, F, G, H, I and J such that, for any pair, a first card placed on top of a second card will produce a hole that is one of the five cards.

Draw the cards.

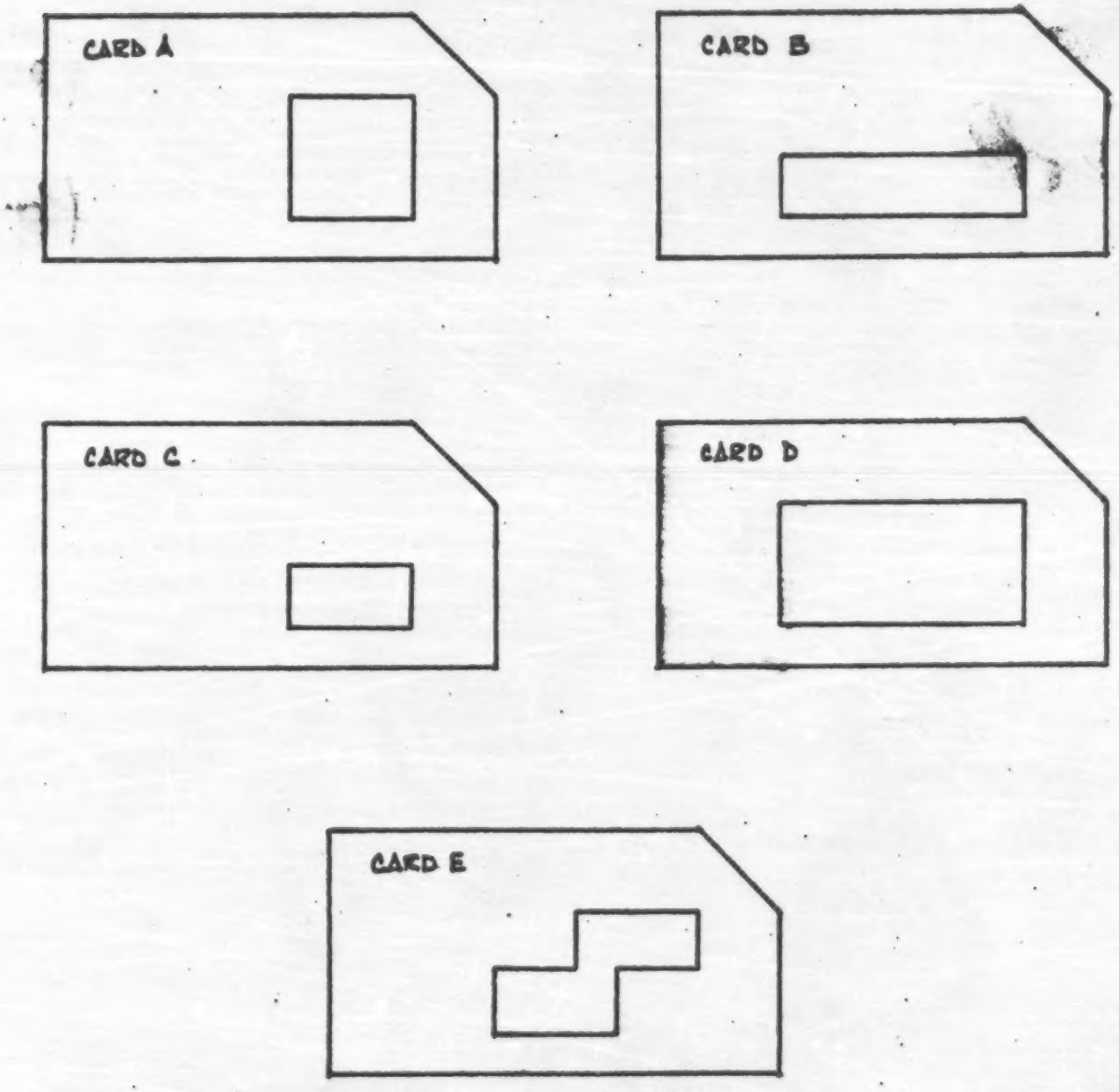

FIGURE 1

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